# PRIMORIAL BASED PRIMES

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(Abstract)
The product of all successive primes starting at Prime[1] = 2 and ending at the mth prime, Prime[m], is called the mth primorial. In order to form primorial based primes an odd number must be added or subtracted from a primorial. This note explores the nature of the primes formed in this process.

The mth primorial is defined as

$$\text{primorial}[m] = \prod_{m=1}^{m} \text{Prime}[n]$$

where Prime[n] is the nth prime number. The primorial is always an even number so that to produce a prime requires adding or subtracting 1 from a primorial or adding and subtracting an odd number from a primorial. Only the latter cases will be considered here. Some obvious properties of such numbers are :

(1.) primorial[m] ± (2 * k + 1) cannot be prime if (2 * k + 1) is less tham Prime[m + 1].

(2.) If (2 * k + 1) is greater than Prime[m] but less than Prime[m] ^ 2 then primorial[m] ± (2 * k + 1 ) can be a prime only if 2 * k + 1 is also a prime.

(3.) For numbers larger than primorial[m] ± Prime[m] ^ 2 primes of the form primorial[m] ± (2 * k + 1) can be prime even if (2 * k + 1) is not a prime.

It is the purpose of this note to point out the properties and advantages of searching for primes of the numbers

N[m, p] = primorial[m] ± Prime [m + p] (1.)

where p varies from 1 to p (max) with p (max) the solution to the equation

Prime[m + p (max)] = Prime[m] ^ 2. (2.)

The principle argument for searching for primes of this form is that primorials with more than a million digits, m = 170 000 for example, can be calculated in a matter of a few seconds with most desk top computers. Hence it is of interest to explore some of the properties of such numbers. In what follows searches for primes in the region N[m, p] with p varied from 1 to p (max) , labeled region 1, will be compared to the numbers

N[m, k] = primorial[m] ± (Prime[m]^2 + 2*k) , (3.)

labeled region 2, where k varies from 1 to Prime[m] (Prime[m] - 1) / 2. Which is the same number of searches required for region 1 if values of (2*k+1) that are not prime are also included in the search.

In order to study the properties of the numbers defined in Eqs. 2 and 3 it is convienent to use the "Mathematica" language where p (max) [m] is defined as

p (max) [m] = PrimePi[Prime[m]^2] - m, (4.)

primorial[m] by,

primorial[m] = Product[Prime[k], {k, 1, m}] (5.)

and the number of digits in the primorial[m] which are given as

numberdigits[m] = Sum[DigitCount[primorial[m], 10, j], {j, 0, 9}]. (6.)

It should be noted that for large primes any number of the form given in Eq. 1 will almost always have the same number of digits as those in the primorial since a prime with 7 to 10 digits will be added or subtracted from a primorial with more than a million digits. Nearly all of the most significant nearly million digits would have to be 9 ' s in the case of adding a prime to a primorial or 1 followed by almost a million zeros in the case of subtracting a prime in order to change the number of digits from that of the primorial.

In order to test for probable primality the following four methods are available in "Mathematica"

PrimeQ[N[m, p]] = True (7.)

NextPrime[N[m, p]] = value of m for the next larger prime (8.)

CompositeQ[N[m, p]] = False (9.)

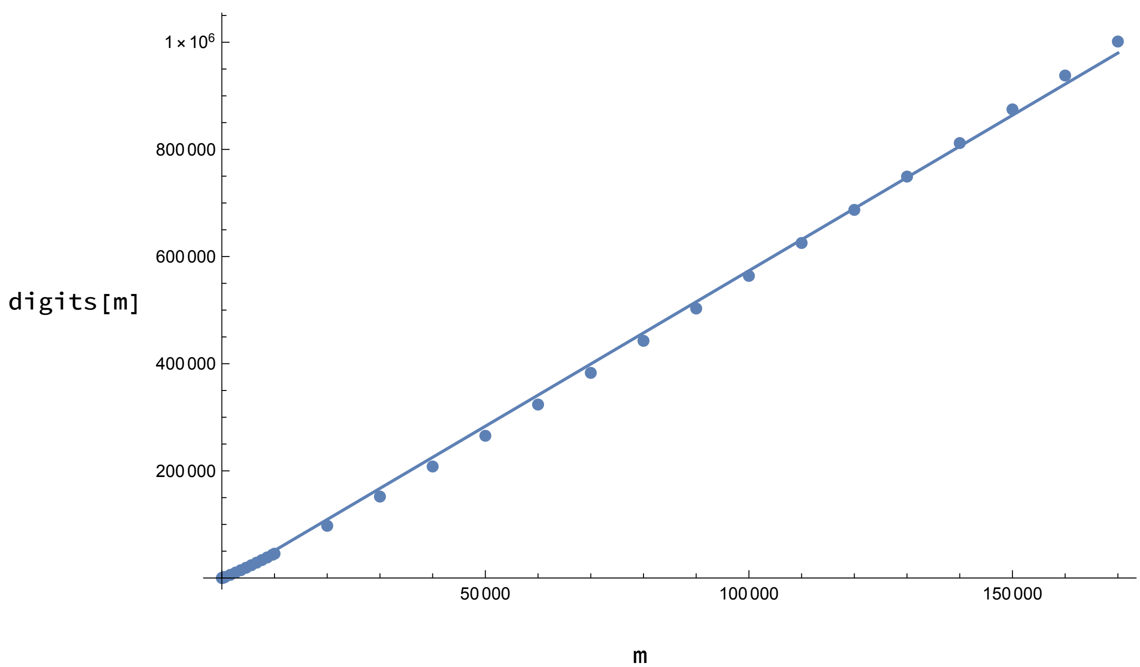
and

PowerMod[2, N[m, p] - 1, N[m, p]] = 1, (10.)

where the latter is Fermat's little theorem. All of these methods require about the same computing time to determine that a number is not prime but Eq. 10 is much faster when the number is probably a prime (about the same time as for not a prime). However, for large primes the time to prove that a number is probably prime requires the use of a super computer although the total number of searches needed to find all of the primes in regions 1 and 2, the number of digits in large primorials, and the primorial itself can be easily computed with a desk top computer. In the remaining section the behavior of the numbers in regions 1 and 2 will be explored for small primes.

In Fig. 1 the dependence of the number of digits in the primorial as a function of m is shown which appears to be nearly a linear function of m.

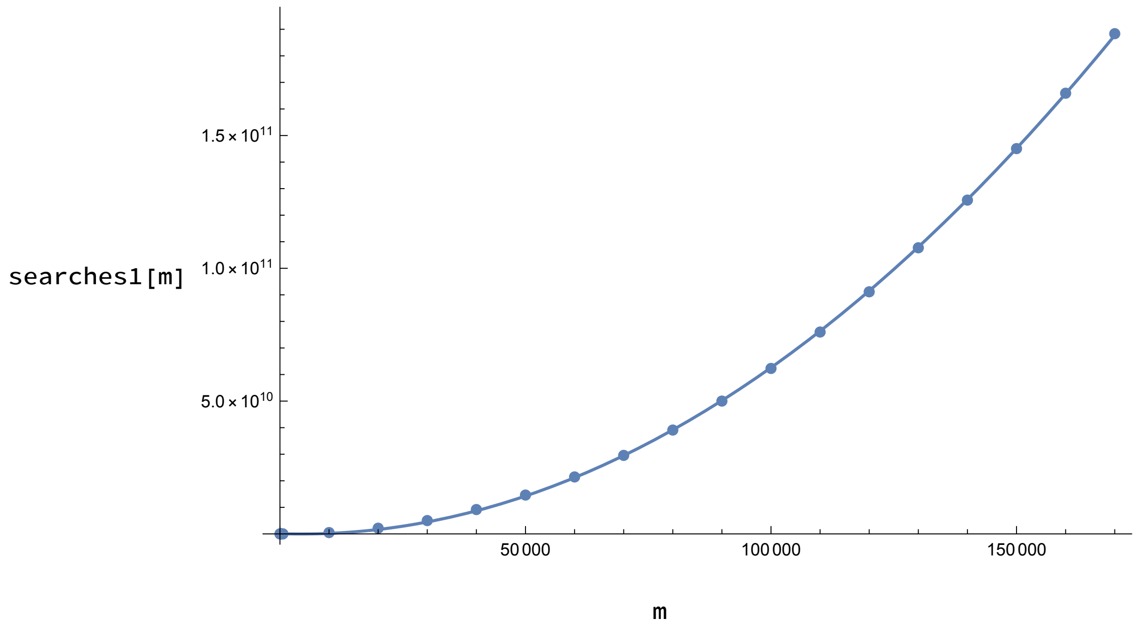
Figure 1 : Number of digits in the mth primorial as a function of m (points). The solid line is the best fit straight line given as : digits[m] = -6888.6 + 5.80435 m .




In Fig. 2 the number of searches, p (max) [m], required to cover all of region 1 is shown as a function of m.

Figure 2 : Number of searches in region 1 required to find all possible primes of the type primorial plus or minus a prime as a function of m (points). The solid line is the best fit quadratic function given as :

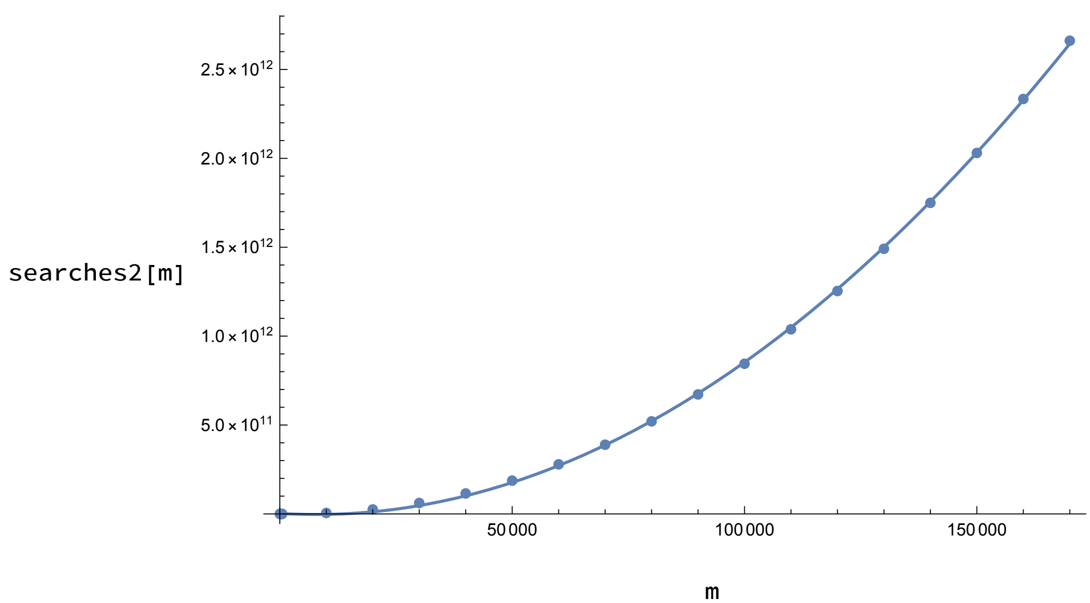
$$\text{searches1}[m] = 9.52099 * 10^7 - 58716\, m + 6.83784\, m^2$$




In Fig. 3 the same plot for the total number of searches required , k (max)[m], to find all primes in region 2 is shown with the best quadratic fit. Note that the number of searches required is more than an order of magnitude larger than for searches in region 1 for numbers with more than a million digits.

Figure 3 : Number of searches in region 2 required to find all possible primes of the type primorial plus an odd number as a function of m (points). The solid line is the best fit quadratic function given as :

$$\text{searches2}[m] = 2.74236 * 10^9 - 1.55020 * 10^6\, m + 100.53557\, m^2$$



The remaining quantities of interest: the number of primes in regions 1 and 2 as a function of m, the prime density as a function of m, and its reciprocal, the average number of searches per prime found, all involve determining probable primality which can only reasonably be determined using a desktop computer for small values of m. In Figures 4 - 7 the dependence of the total number of primes found in complete searches of regions 1 and 2 are shown along with best linear fits of the data for values of m up to 500.

Figure 4: Total number of primes found in region 1 of the type primorial plus a prime number as a function of m (points). The solid line is the best fit to a linear function given as: prime1p[m] = -162.3895 + 7.18096 m

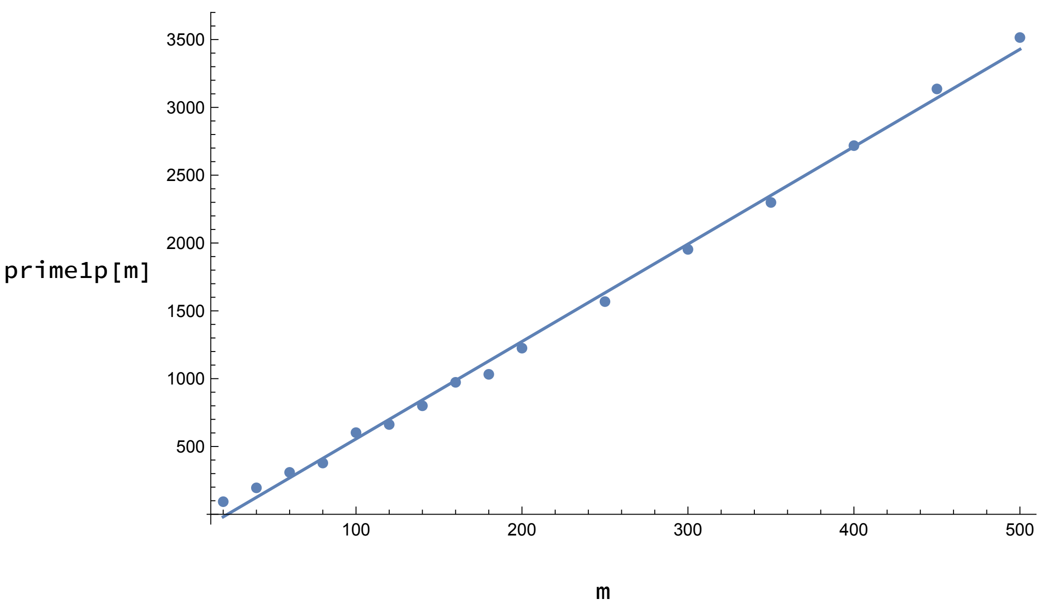



Figure 5: Total number of primes found in region 1 of the type primorial minus a prime number as a function of m (points). The solid line is the best fit to a linear function given as:

prime1m[m] = -139.134797 + 6.99766 m

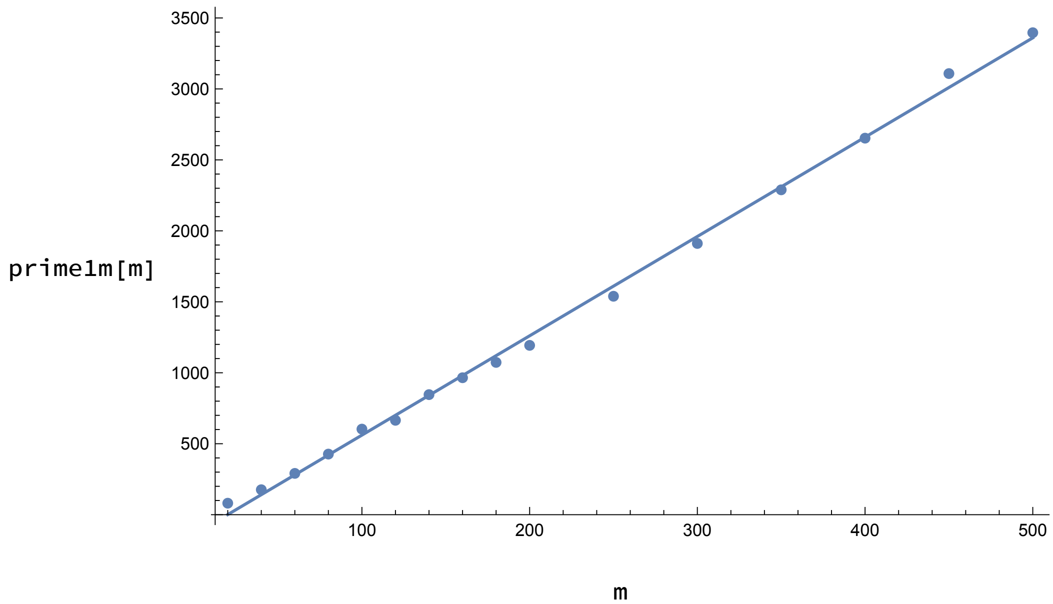



Figure 6 : Total number of primes found in region 2 of the type primorial plus all odd numbers as a function of m (points) . The solid line is the best fit to a linear function given as : prime2p[m] = -178.9333 + 6.91721 m

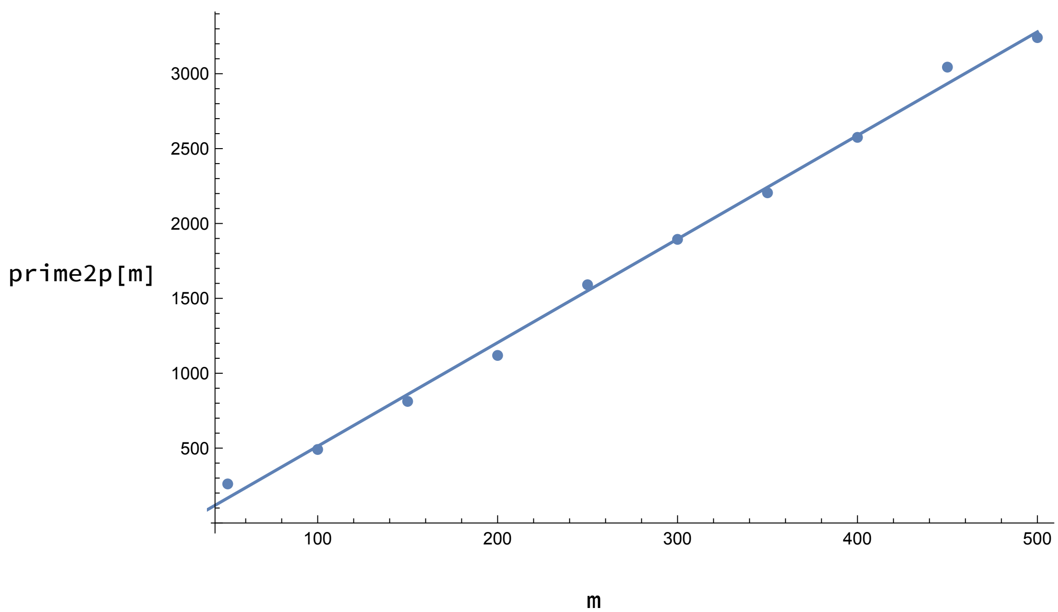
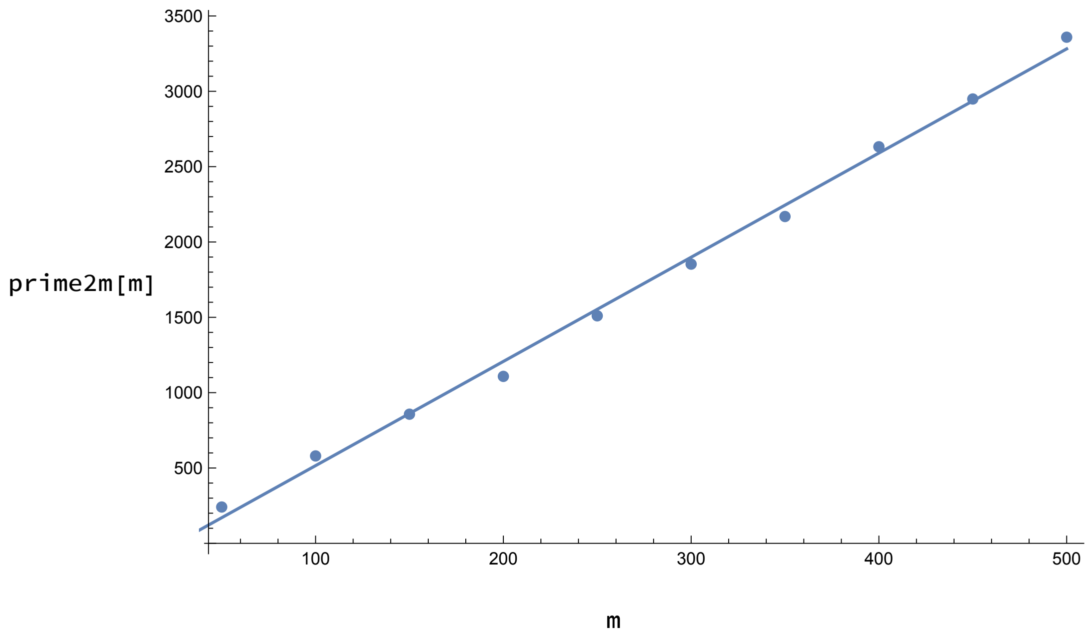



Figure 7 : Total number of primes found in region 2 of the type primorial minus all odd numbers as a function of m (points) . The solid line is the best fit to a linear function given as : prime2m[m] = -175.73333 + 6.914667 m




It should be noted that the total number of primes in region 1 is greater than the number in region 2 even if the numbers with Prime[m]^2 + 2*k not prime are included.

In Fig. 8 the average number of searches to find a primorial plus a prime in region 1 is displayed. Extrapolation of this result to m = 170 000 (million digit primes) suggests that, on average, about 81 000 searches would be required to locate a probable prime of that size. Fig. 9 displays the results for a primorial minus a prime as a function of m with a projected number of searches to find a prime of about 84 000.

Figure 8 : Average number of searches to find a prime in region 1 of the type primorial plus a prime as a function of m (points). The solid line is the best fit to a linear function given as : search/primep[m] = -1.67901 + 0.474315 m

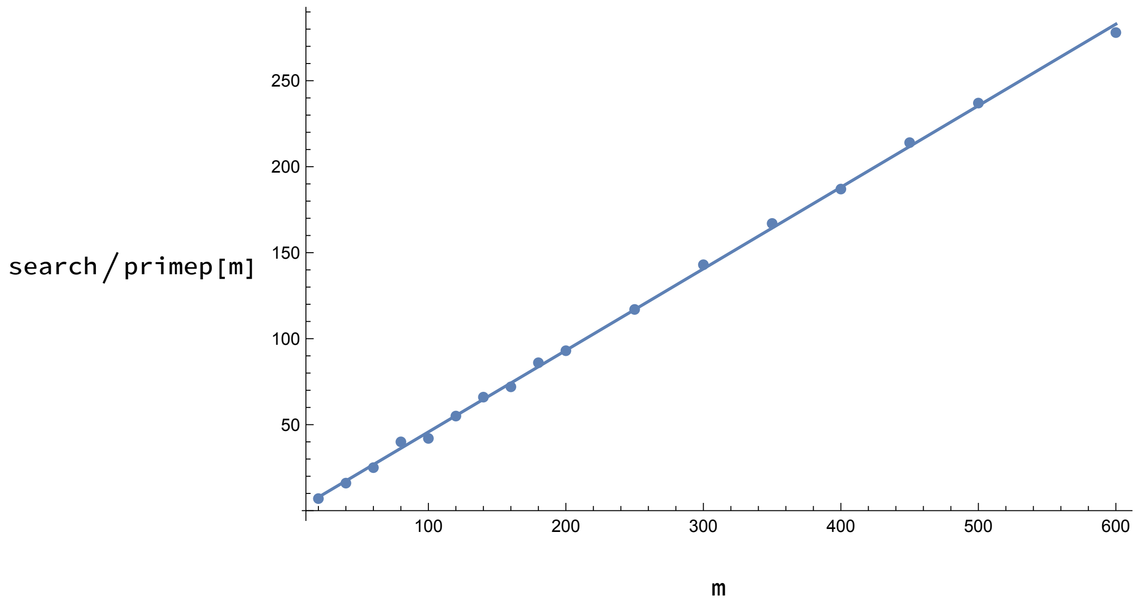
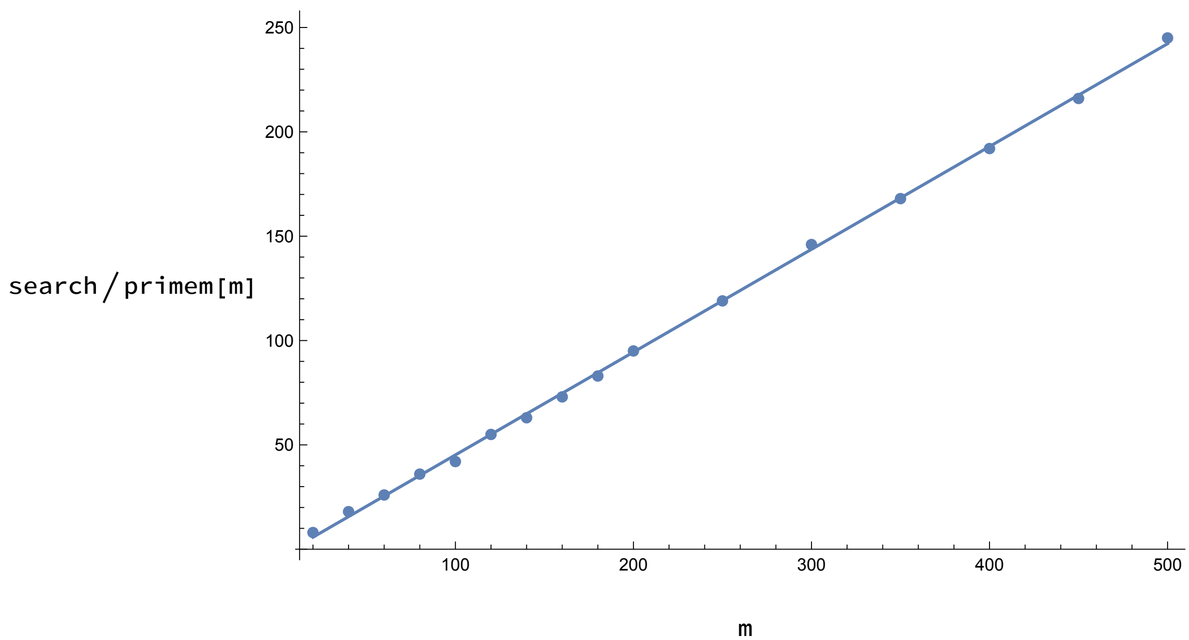



Figure 9 : Average number of searches to find a prime in region 1 of the type primorial minus a prime as a function of m (points). The solid line is the best fit to a linear function given as : search / primem[m] = -4.08821 + 0.492660 m




In Figures 10 and 11 the average number of searches to find a prime in region 2 are shown. For m = 170 000 the extrapolated number of searches is more than 670 000 or approximately 8 times the number of searches required for region 1.

Figure 10 : Average number of searches to find a prime in region 2 of the type primorial plus all odd numbers as a function of m (points). The solid line is the best fit to a linear function given as : search / primep2[m] = -109.19498 + 3.948692 m

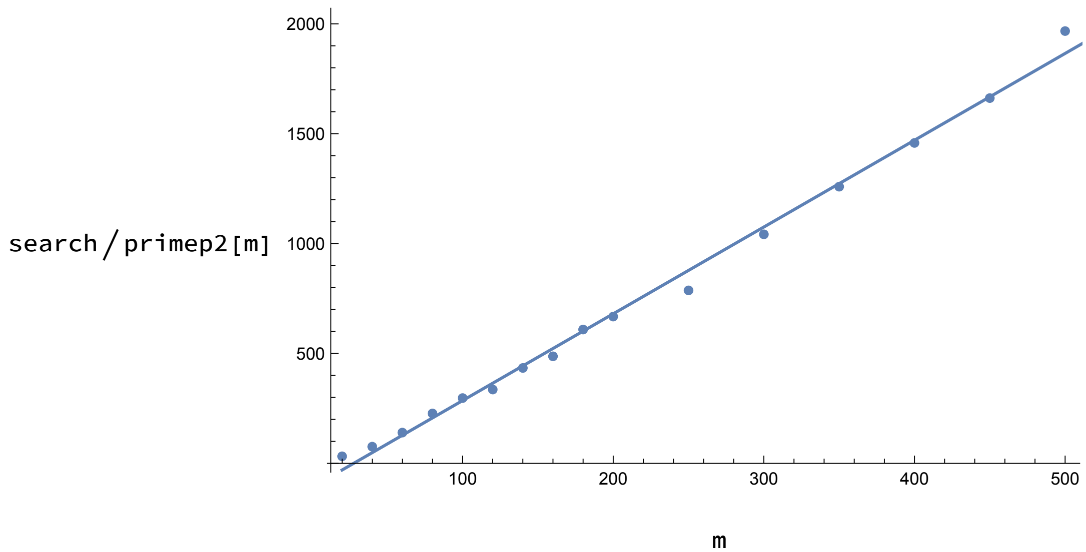
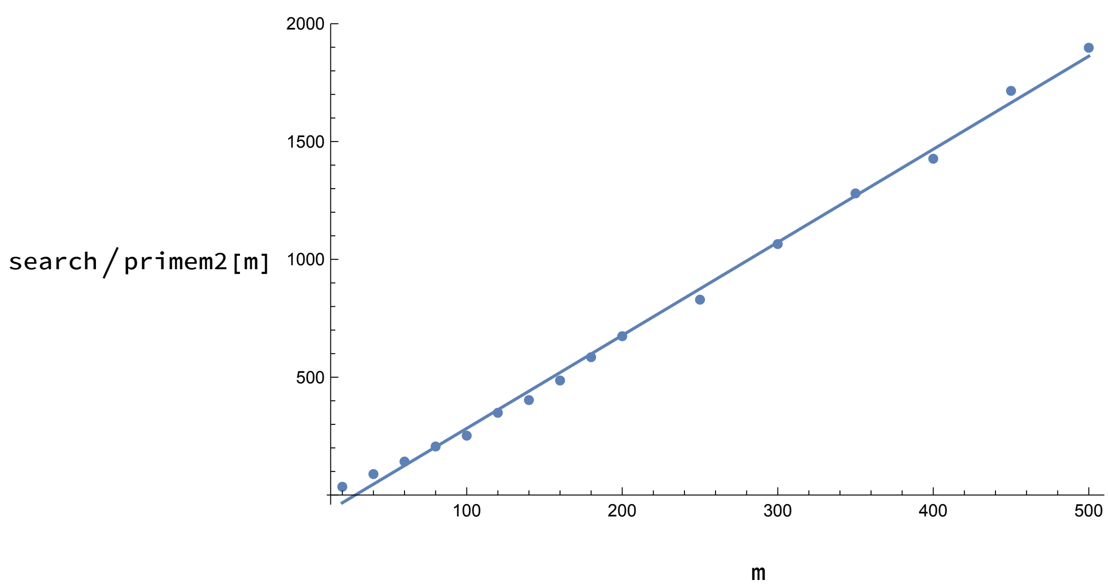



Figure 11 : Average number of searches to find a prime in region 2 of the type primorial minus all odd numbers as a function of m (points). The solid line is the best fit to a linear function given as : search / primem2[m] - 111.94147 + 3.948079 m




In summary, it has been shown that the number of searches required to find all primes in regions 1 and 2 is approximately a quadratic function of m where m is the mth primorial. The number of digits in the primorial the number of probable primes in regions 1 and 2 , and the average number of searches required to find a probable prime in regions 1 and 2 are all approximately linear functions of m.